Integration Recurrence Equations for $\int\left(\sin^j(z)\right)^m\left(A+B\sin^k(z)+C\sin^{2k}(z)\right)\left(a+b\sin^k(z)\right)^n\,dz$ when $j^2=1\bigwedge k^2=1\bigwedge a^2\neq b^2$

- **Recurrence 1: If $j^2 = k^2 = 1$, then**

$$2b(n+1)\left(a^2-b^2\right)\int\left(\mathrm{Sin}[z]^j\right)^m\left(A+B\,\mathrm{Sin}[z]^k+C\,\mathrm{Sin}[z]^{2k}\right)\left(a+b\,\mathrm{Sin}[z]^k\right)^n dz =$$
$$-2\left(a^2C-abB+b^2A\right)\mathrm{Cos}[z]\left(\mathrm{Sin}[z]^j\right)^m\left(a+b\,\mathrm{Sin}[z]^k\right)^{n+1}+$$
$$\int\left(\mathrm{Sin}[z]^j\right)^{m-jk}\left(\left(a^2C-abB+b^2A\right)(2jkm+k-1)+2b(n+1)(a(C+A)-bB)\,\mathrm{Sin}[z]^k-\right.$$
$$\left.\left(2\left(b^2A-abB+b^2C\right)(n+1)+\left(a^2C-abB+b^2A\right)(2jkm+k+1)\right)\mathrm{Sin}[z]^{2k}\right)\left(a+b\,\mathrm{Sin}[z]^k\right)^{n+1}dz$$

- **Recurrence 2: If $j^2 = k^2 = 1$, then**

$$b(2jkm+2n+k+3)\int\left(\mathrm{Sin}[z]^j\right)^m\left(A+B\,\mathrm{Sin}[z]^k+C\,\mathrm{Sin}[z]^{2k}\right)\left(a+b\,\mathrm{Sin}[z]^k\right)^n dz =$$
$$-2C\,\mathrm{Cos}[z]\left(\mathrm{Sin}[z]^j\right)^m\left(a+b\,\mathrm{Sin}[z]^k\right)^{n+1}+$$
$$\int\left(\mathrm{Sin}[z]^j\right)^{m-jk}\left(aC(2jkm+k-1)+b(2A+(A+C)(2jkm+2n+k+1))\,\mathrm{Sin}[z]^k+\right.$$
$$\left.(2bB(n+1)+(bB-aC)(2jkm+k+1))\,\mathrm{Sin}[z]^{2k}\right)\left(a+b\,\mathrm{Sin}[z]^k\right)^n dz$$

- **Recurrence 3: If $j^2 = k^2 = 1$, then**

$$(2jkm+2n+k+3)\int\left(\mathrm{Sin}[z]^j\right)^m\left(A+B\,\mathrm{Sin}[z]^k+C\,\mathrm{Sin}[z]^{2k}\right)\left(a+b\,\mathrm{Sin}[z]^k\right)^n dz =$$
$$-2C\,\mathrm{Cos}[z]\left(\mathrm{Sin}[z]^j\right)^{m+jk}\left(a+b\,\mathrm{Sin}[z]^k\right)^n+$$
$$\int\left(\mathrm{Sin}[z]^j\right)^m$$
$$\left(a(2A(n+1)+(A+C)(2jkm+k+1))+(2bA+2aB+(bA+aB+bC)(2jkm+2n+k+1))\,\mathrm{Sin}[z]^k+\right.$$
$$\left.(2aCn+bB(2jkm+2n+k+3))\,\mathrm{Sin}[z]^{2k}\right)\left(a+b\,\mathrm{Sin}[z]^k\right)^{n-1}dz$$

- **Recurrence 4: If $j^2 = k^2 = 1$, then**

$$(2jkm+k+1)\int\left(\mathrm{Sin}[z]^j\right)^m\left(A+B\,\mathrm{Sin}[z]^k+C\,\mathrm{Sin}[z]^{2k}\right)\left(a+b\,\mathrm{Sin}[z]^k\right)^n dz =$$
$$2A\,\mathrm{Cos}[z]\left(\mathrm{Sin}[z]^j\right)^{m+jk}\left(a+b\,\mathrm{Sin}[z]^k\right)^n+$$
$$\int\left(\mathrm{Sin}[z]^j\right)^{m+jk}\left(aB(2jkm+k+1)-2bAn+(2aA+(aA+aC+bB)(2jkm+k+1))\,\mathrm{Sin}[z]^k+\right.$$
$$\left.b(2A(n+1)+(A+C)(2jkm+k+1))\,\mathrm{Sin}[z]^{2k}\right)\left(a+b\,\mathrm{Sin}[z]^k\right)^{n-1}dz$$

- **Recurrence 5: If $j^2 = k^2 = 1$, then**

$$a\,(2\,j\,k\,m+k+1)\int\left(\mathrm{Sin}[z]^j\right)^m\left(A+B\,\mathrm{Sin}[z]^k+C\,\mathrm{Sin}[z]^{2\,k}\right)\left(a+b\,\mathrm{Sin}[z]^k\right)^n dz =$$
$$2\,A\,\mathrm{Cos}[z]\left(\mathrm{Sin}[z]^j\right)^{m+j\,k}\left(a+b\,\mathrm{Sin}[z]^k\right)^{n+1}+$$
$$\int\left(\mathrm{Sin}[z]^j\right)^{m+j\,k}\Big(((a\,B-b\,A)\,(2\,j\,k\,m+k+1)-2\,b\,A\,(n+1)+a\,(2\,A+(A+C)\,(2\,j\,k\,m+k+1)))\,\mathrm{Sin}[z]^k+$$
$$b\,A\,(2\,j\,k\,m+2\,n+k+5)\,\mathrm{Sin}[z]^{2\,k}\Big)\left(a+b\,\mathrm{Sin}[z]^k\right)^n dz$$

- **Recurrence 6: If $j^2 = k^2 = 1$, then**

$$2\,a\,(n+1)\left(a^2-b^2\right)\int\left(\mathrm{Sin}[z]^j\right)^m\left(A+B\,\mathrm{Sin}[z]^k+C\,\mathrm{Sin}[z]^{2\,k}\right)\left(a+b\,\mathrm{Sin}[z]^k\right)^n dz =$$
$$2\left(b^2\,A-a\,b\,B+a^2\,C\right)\mathrm{Cos}[z]\left(\mathrm{Sin}[z]^j\right)^{m+j\,k}\left(a+b\,\mathrm{Sin}[z]^k\right)^{n+1}+$$
$$\int\left(\mathrm{Sin}[z]^j\right)^m$$
$$\Big(2\,A\left(a^2-b^2\right)(n+1)-\left(a^2\,C-a\,b\,B+b^2\,A\right)(2\,j\,k\,m+k+1)-2\,a\,(b\,A-a\,B+b\,C)\,(n+1)\,\mathrm{Sin}[z]^k+$$
$$\left(b^2\,A-a\,b\,B+a^2\,C\right)(2\,j\,k\,m+2\,n+k+5)\,\mathrm{Sin}[z]^{2\,k}\Big)\left(a+b\,\mathrm{Sin}[z]^k\right)^{n+1} dz$$

Integration Recurrence Equations for $\int(\sin^j(z))^m\,(A+B\sin^k(z)+C\sin^{2k}(z))\,(a+b\sin^k(z))^n\,dz$ when $j^2=1 \bigwedge k^2=1 \bigwedge a^2=b^2$

- **Recurrence 7: If $j^2 = k^2 = 1 \wedge a^2 - b^2 = 0$, then**

$$2a^2(2n+1)\int(\mathrm{Sin}[z]^j)^m\,(A+B\,\mathrm{Sin}[z]^k)\,(a+b\,\mathrm{Sin}[z]^k)^n\,dz =$$
$$2a(bA-aB)\,\mathrm{Cos}[z]\,(\mathrm{Sin}[z]^j)^m\,(a+b\,\mathrm{Sin}[z]^k)^n +$$
$$\int(\mathrm{Sin}[z]^j)^{m-jk}\,\big(-(bA-aB)(2jkm+k-1)+$$
$$(2(bBn+aA(n+1))+(aA-bB)(2jkm+k-1))\,\mathrm{Sin}[z]^k\big)\,(a+b\,\mathrm{Sin}[z]^k)^{n+1}\,dz$$

- **Recurrence 8: If $j^2 = k^2 = 1 \wedge a^2 - b^2 = 0$, then**

$$a(2jkm+2n+k+1)\int(\mathrm{Sin}[z]^j)^m\,(A+B\,\mathrm{Sin}[z]^k)\,(a+b\,\mathrm{Sin}[z]^k)^n\,dz =$$
$$-2aB\,\mathrm{Cos}[z]\,(\mathrm{Sin}[z]^j)^m\,(a+b\,\mathrm{Sin}[z]^k)^n +$$
$$\int(\mathrm{Sin}[z]^j)^{m-jk}\,\big(aB(2jkm+k-1)+(2bBn+aA(2jkm+2n+k+1))\,\mathrm{Sin}[z]^k\big)\,(a+b\,\mathrm{Sin}[z]^k)^n\,dz$$

- **Recurrence 9: If $j^2 = k^2 = 1 \wedge a^2 - b^2 = 0$, then**

$$(2jkm+2n+k+1)\int(\mathrm{Sin}[z]^j)^m\,(A+B\,\mathrm{Sin}[z]^k)\,(a+b\,\mathrm{Sin}[z]^k)^n\,dz =$$
$$-2bB\,\mathrm{Cos}[z]\,(\mathrm{Sin}[z]^j)^{m+jk}\,(a+b\,\mathrm{Sin}[z]^k)^{n-1} +$$
$$\int(\mathrm{Sin}[z]^j)^m\,\big(2aAn+(aA+bB)(2jkm+k+1)+$$
$$(2(bA+aBn)+(bA+aB)(2jkm+2n+k-1))\,\mathrm{Sin}[z]^k\big)\,(a+b\,\mathrm{Sin}[z]^k)^{n-1}\,dz$$

- **Recurrence 10: If $j^2 = k^2 = 1 \wedge a^2 - b^2 = 0$, then**

$$(2jkm+k+1)\int(\mathrm{Sin}[z]^j)^m\,(A+B\,\mathrm{Sin}[z]^k)\,(a+b\,\mathrm{Sin}[z]^k)^n\,dz =$$
$$2aA\,\mathrm{Cos}[z]\,(\mathrm{Sin}[z]^j)^{m+jk}\,(a+b\,\mathrm{Sin}[z]^k)^{n-1} +$$
$$\int(\mathrm{Sin}[z]^j)^{m+jk}\,\big((bA+aB)(2jkm+k+1)-2bA(n-1)+(2aAn+(aA+bB)(2jkm+k+1))\,\mathrm{Sin}[z]^k\big)$$
$$(a+b\,\mathrm{Sin}[z]^k)^{n-1}\,dz$$

- **Recurrence 11: If $j^2 = k^2 = 1 \wedge a^2 - b^2 = 0$, then**

$$a(2jkm+k+1)\int(\mathrm{Sin}[z]^j)^m\,(A+B\,\mathrm{Sin}[z]^k)\,(a+b\,\mathrm{Sin}[z]^k)^n\,dz =$$
$$2aA\,\mathrm{Cos}[z]\,(\mathrm{Sin}[z]^j)^{m+jk}\,(a+b\,\mathrm{Sin}[z]^k)^n +$$
$$\int(\mathrm{Sin}[z]^j)^{m+jk}\,\big(aB(2jkm+k+1)-2bAn+aA(2jkm+2n+k+3)\,\mathrm{Sin}[z]^k\big)\,(a+b\,\mathrm{Sin}[z]^k)^n\,dz$$

Recurrence 12: If $j^2 = k^2 = 1 \wedge a^2 - b^2 = 0$, then

$$2\,a^2\,(2\,n+1)\int\left(\mathrm{Sin}[z]^j\right)^m\left(A+B\,\mathrm{Sin}[z]^k\right)\left(a+b\,\mathrm{Sin}[z]^k\right)^n\,dz =$$

$$-2\,b\,(b\,A-a\,B)\,\mathrm{Cos}[z]\left(\mathrm{Sin}[z]^j\right)^{m+j\,k}\left(a+b\,\mathrm{Sin}[z]^k\right)^n +$$

$$\int\left(\mathrm{Sin}[z]^j\right)^m\left(2\,a\,A\,(2\,n+1)+(a\,A-b\,B)\,(2\,j\,k\,m+k+1)-(b\,A-a\,B)\,(2\,j\,k\,m+2\,n+k+3)\,\mathrm{Sin}[z]^k\right)$$

$$\left(a+b\,\mathrm{Sin}[z]^k\right)^{n+1}\,dz$$