Error Function Integration Problem 1

$$\int x^m \text{Erf}[b\,x]^2\,dx$$

- *Rubi* is able to integrate $x^m\,\text{Erf}[b\,x]^2$ for odd m except -1:

```
Int[x^3 Erf[b x]^2, x]
```

$$\frac{e^{-2 b^2 x^2}}{2 b^4 \pi}+\frac{e^{-2 b^2 x^2} x^2}{4 b^2 \pi}+\frac{e^{-b^2 x^2} x \left(3+2 b^2 x^2\right) \text{Erf}[b x]}{4 b^3 \sqrt{\pi}}-\frac{3 \text{Erf}[b x]^2}{16 b^4}+\frac{1}{4} x^4 \text{Erf}[b x]^2$$

```
Int[x Erf[b x]^2, x]
```

$$\frac{e^{-2 b^2 x^2}}{2 b^2 \pi}+\frac{e^{-b^2 x^2} x \text{Erf}[b x]}{b \sqrt{\pi}}-\frac{\text{Erf}[b x]^2}{4 b^2}+\frac{1}{2} x^2 \text{Erf}[b x]^2$$

```
Int[Erf[b x]^2/x, x]
```

$$\text{Int}\left[\frac{\text{Erf}[b x]^2}{x},x\right]$$

```
Int[Erf[b x]^2/x^3, x]
```

$$-\frac{2 b e^{-b^2 x^2} \text{Erf}[b x]}{\sqrt{\pi} x}-b^2 \text{Erf}[b x]^2-\frac{\text{Erf}[b x]^2}{2 x^2}+\frac{2 b^2 \text{ExpIntegralEi}\left[-2 b^2 x^2\right]}{\pi}$$

```
Int[Erf[b x]^2/x^5, x]
```

$$-\frac{b^2 e^{-2 b^2 x^2}}{3 \pi x^2}-\frac{b e^{-b^2 x^2} \left(1-2 b^2 x^2\right) \text{Erf}[b x]}{3 \sqrt{\pi} x^3}+\frac{1}{3} b^4 \text{Erf}[b x]^2-\frac{\text{Erf}[b x]^2}{4 x^4}-\frac{4 b^4 \text{ExpIntegralEi}\left[-2 b^2 x^2\right]}{3 \pi}$$

- *Mathematica* is not able to integrate $x^m\,\text{Erf}[b\,x]^2$ for negative odd m:

$$\int x^3\, \text{Erf}[b x]^2\, dx$$

$$\frac{e^{-2 b^2 x^2} \left(8+4 b^2 x^2+4 b e^{b^2 x^2} \sqrt{\pi} x \left(3+2 b^2 x^2\right) \text{Erf}[b x]+e^{2 b^2 x^2} \pi \left(-3+4 b^4 x^4\right) \text{Erf}[b x]^2\right)}{16 b^4 \pi}$$

$$\int x\, \text{Erf}[b x]^2\, dx$$

$$\frac{2 e^{-2 b^2 x^2}+4 b e^{-b^2 x^2} \sqrt{\pi} x \text{Erf}[b x]+\pi \left(-1+2 b^2 x^2\right) \text{Erf}[b x]^2}{4 b^2 \pi}$$

$$\int \frac{\mathrm{Erf}[b\,x]^2}{x}\,dx$$

$$\int \frac{\mathrm{Erf}[b\,x]^2}{x}\,dx$$

$$\int \frac{\mathrm{Erf}[b\,x]^2}{x^3}\,dx$$

$$\int \frac{\mathrm{Erf}[b\,x]^2}{x^3}\,dx$$

$$\int \frac{\mathrm{Erf}[b\,x]^2}{x^5}\,dx$$

$$\int \frac{\mathrm{Erf}[b\,x]^2}{x^5}\,dx$$

- *Maple* is not able to integrate $x^m\,\mathrm{Erf}[b\,x]^2$ for any odd m:

```
int(x^3*erf(b*x)^2, x);
```

$$\int x^3\,\mathrm{erf}(b\,x)^2\,dx$$

```
int(x*erf(b*x)^2, x);
```

$$\int x\,\mathrm{erf}(b\,x)^2\,dx$$

```
int(erf(b*x)^2/x, x);
```

$$\int \frac{\mathrm{erf}(b\,x)^2}{x}\,dx$$

```
int(erf(b*x)^2/x^3, x);
```

$$\int \frac{\mathrm{erf}(b\,x)^2}{x^3}\,dx$$

```
int(erf(b*x)^2/x^5, x);
```

$$\int \frac{\mathrm{erf}(b\,x)^2}{x^5}\,dx$$

Note that these systems give similar results to the above for the complementary and imaginary error functions.

Error Function Integration Problem 2

$$\int x^m \operatorname{Erf}[a+b\,x]^2\,dx$$

- *Rubi* is able to integrate $x^m \operatorname{Erf}[a+b\,x]^2$ for integer $m \geq 0$:

```
Int[Erf[a + b x]^2, x]
```

$$\frac{2\,e^{-(a+b\,x)^2}\operatorname{Erf}[a+b\,x]}{b\sqrt{\pi}}+\frac{(a+b\,x)\operatorname{Erf}[a+b\,x]^2}{b}-\frac{\sqrt{\frac{2}{\pi}}\operatorname{Erf}\left[\sqrt{2}\,(a+b\,x)\right]}{b}$$

```
Int[x Erf[a + b x]^2, x]
```

$$\frac{e^{-2\,(a+b\,x)^2}}{2\,b^2\,\pi}-\frac{e^{-(a+b\,x)^2}\,(a-b\,x)\operatorname{Erf}[a+b\,x]}{b^2\sqrt{\pi}}-\frac{\left(1+2\,a^2-2\,b^2\,x^2\right)\operatorname{Erf}[a+b\,x]^2}{4\,b^2}+\frac{a\sqrt{\frac{2}{\pi}}\operatorname{Erf}\left[\sqrt{2}\,(a+b\,x)\right]}{b^2}$$

```
Int[x^2 Erf[a + b x]^2, x]
```

$$-\frac{2\,a\,e^{-2\,(a+b\,x)^2}}{3\,b^3\,\pi}+\frac{e^{-2\,(a+b\,x)^2}\,x}{3\,b^2\,\pi}+\frac{2\,e^{-(a+b\,x)^2}\left(1+a^2-a\,b\,x+b^2\,x^2\right)\operatorname{Erf}[a+b\,x]}{3\,b^3\sqrt{\pi}}+$$
$$\frac{\left(3\,a+2\,a^3+2\,b^3\,x^3\right)\operatorname{Erf}[a+b\,x]^2}{6\,b^3}-\frac{\left(5+12\,a^2\right)\operatorname{Erf}\left[\sqrt{2}\,(a+b\,x)\right]}{6\,b^3\sqrt{2\,\pi}}$$

- *Mathematica* is unable integrate $x^m \operatorname{Erf}[a+b\,x]^2$ for integer $m > 0$:

$$\int \operatorname{Erf}[a+b\,x]^2\,dx$$

$$\frac{\frac{2\,e^{-(a+b\,x)^2}\operatorname{Erf}[a+b\,x]}{\sqrt{\pi}}+(a+b\,x)\operatorname{Erf}[a+b\,x]^2-\sqrt{\frac{2}{\pi}}\operatorname{Erf}\left[\sqrt{2}\,(a+b\,x)\right]}{b}$$

$$\int x\operatorname{Erf}[a+b\,x]^2\,dx$$

$$\int x\operatorname{Erf}[a+b\,x]^2\,dx$$

$$\int x^2\operatorname{Erf}[a+b\,x]^2\,dx$$

$$\int x^2\operatorname{Erf}[a+b\,x]^2\,dx$$

- *Maple* is unable integrate $x^m\,\mathrm{Erf}[a+b\,x]^2$ for integer $m>0$:

```
int(erf(a+b*x)^2, x);
```

$$\frac{\frac{2\,e^{-(a+b\,x)^2}\,\mathrm{Erf}[a+b\,x]}{\sqrt{\pi}}+(a+b\,x)\,\mathrm{Erf}[a+b\,x]^2-\sqrt{\frac{2}{\pi}}\,\mathrm{Erf}\left[\sqrt{2}\,(a+b\,x)\right]}{b}$$

```
int(x*erf(a+b*x)^2, x);
```

$$\int x\,\mathrm{Erf}[a+b\,x]^2\,dx$$

```
int(x^2*erf(a+b*x)^2, x);
```

$$\int x^2\,\mathrm{Erf}[a+b\,x]^2\,dx$$

Note that these systems give similar results to the above for the complementary and imaginary error functions.

Error Function Integration Problem 3

$$\int x^m \,\mathrm{FresnelS}[b\,x]^2\,dx$$

- *Rubi* is able to integrate x^m FresnelS[b x]^2 if m mod 4 equals 3 except if m equals -1:

```
Int[x^7 FresnelS[b x]^2, x]
```

$$-\frac{105\,x^2}{16\,b^6\,\pi^4}+\frac{7\,x^6}{48\,b^2\,\pi^2}-\frac{55\,x^2\,\mathrm{Cos}\left[b^2\,\pi\,x^2\right]}{16\,b^6\,\pi^4}+\frac{x^6\,\mathrm{Cos}\left[b^2\,\pi\,x^2\right]}{16\,b^2\,\pi^2}-\frac{35\,x^3\,\mathrm{Cos}\left[\frac{1}{2}\,b^2\,\pi\,x^2\right]\,\mathrm{FresnelS}[b\,x]}{4\,b^5\,\pi^3}+\frac{x^7\,\mathrm{Cos}\left[\frac{1}{2}\,b^2\,\pi\,x^2\right]\,\mathrm{FresnelS}[b\,x]}{4\,b\,\pi}-\frac{105\,\mathrm{FresnelS}[b\,x]^2}{8\,b^8\,\pi^4}+\frac{1}{8}\,x^8\,\mathrm{FresnelS}[b\,x]^2+\frac{105\,x\,\mathrm{FresnelS}[b\,x]\,\mathrm{Sin}\left[\frac{1}{2}\,b^2\,\pi\,x^2\right]}{4\,b^7\,\pi^4}-\frac{7\,x^5\,\mathrm{FresnelS}[b\,x]\,\mathrm{Sin}\left[\frac{1}{2}\,b^2\,\pi\,x^2\right]}{4\,b^3\,\pi^2}+\frac{10\,\mathrm{Sin}\left[b^2\,\pi\,x^2\right]}{b^8\,\pi^5}-\frac{5\,x^4\,\mathrm{Sin}\left[b^2\,\pi\,x^2\right]}{8\,b^4\,\pi^3}$$

```
Int[x^3 FresnelS[b x]^2, x]
```

$$\frac{3\,x^2}{8\,b^2\,\pi^2}+\frac{x^2\,\mathrm{Cos}\left[b^2\,\pi\,x^2\right]}{8\,b^2\,\pi^2}+\frac{x^3\,\mathrm{Cos}\left[\frac{1}{2}\,b^2\,\pi\,x^2\right]\,\mathrm{FresnelS}[b\,x]}{2\,b\,\pi}+\frac{3\,\mathrm{FresnelS}[b\,x]^2}{4\,b^4\,\pi^2}+\frac{1}{4}\,x^4\,\mathrm{FresnelS}[b\,x]^2-\frac{3\,x\,\mathrm{FresnelS}[b\,x]\,\mathrm{Sin}\left[\frac{1}{2}\,b^2\,\pi\,x^2\right]}{2\,b^3\,\pi^2}-\frac{\mathrm{Sin}\left[b^2\,\pi\,x^2\right]}{2\,b^4\,\pi^3}$$

```
Int[FresnelS[b x]^2/x, x]
```

$$\mathrm{Int}\left[\frac{\mathrm{FresnelS}[b\,x]^2}{x},\,x\right]$$

```
Int[FresnelS[b x]^2/x^5, x]
```

$$-\frac{b^2}{24\,x^2}+\frac{b^2\,\mathrm{Cos}\left[b^2\,\pi\,x^2\right]}{24\,x^2}-\frac{b^3\,\pi\,\mathrm{Cos}\left[\frac{1}{2}\,b^2\,\pi\,x^2\right]\,\mathrm{FresnelS}[b\,x]}{6\,x}-\frac{1}{12}\,b^4\,\pi^2\,\mathrm{FresnelS}[b\,x]^2-\frac{\mathrm{FresnelS}[b\,x]^2}{4\,x^4}-\frac{b\,\mathrm{FresnelS}[b\,x]\,\mathrm{Sin}\left[\frac{1}{2}\,b^2\,\pi\,x^2\right]}{6\,x^3}+\frac{1}{12}\,b^4\,\pi\,\mathrm{SinIntegral}\left[b^2\,\pi\,x^2\right]$$

```
Int[FresnelS[b x]^2/x^9, x]
```

$$-\frac{b^2}{336 x^6}+\frac{b^6 \pi^2}{1680 x^2}+\frac{b^2 \operatorname{Cos}\left[b^2 \pi x^2\right]}{336 x^6}-\frac{b^6 \pi^2 \operatorname{Cos}\left[b^2 \pi x^2\right]}{336 x^2}-\frac{b^3 \pi \operatorname{Cos}\left[\frac{1}{2} b^2 \pi x^2\right] \operatorname{FresnelS}[b x]}{140 x^5}+$$
$$\frac{b^7 \pi^3 \operatorname{Cos}\left[\frac{1}{2} b^2 \pi x^2\right] \operatorname{FresnelS}[b x]}{420 x}+\frac{1}{840} b^8 \pi^4 \operatorname{FresnelS}[b x]^2-\frac{\operatorname{FresnelS}[b x]^2}{8 x^8}-\frac{b \operatorname{FresnelS}[b x] \operatorname{Sin}\left[\frac{1}{2} b^2 \pi x^2\right]}{28 x^7}+$$
$$\frac{b^5 \pi^2 \operatorname{FresnelS}[b x] \operatorname{Sin}\left[\frac{1}{2} b^2 \pi x^2\right]}{420 x^3}-\frac{b^4 \pi \operatorname{Sin}\left[b^2 \pi x^2\right]}{420 x^4}-\frac{1}{280} b^8 \pi^3 \operatorname{SinIntegral}\left[b^2 \pi x^2\right]$$

- *Mathematica* is not able to integrate $x^m \operatorname{FresnelS}[b x]^2$ if m mod 4 equals 3:

$$\int x^7 \operatorname{FresnelS}[b x]^2 \, dx$$

$$\int x^7 \operatorname{FresnelS}[b x]^2 \, dx$$

$$\int x^3 \operatorname{FresnelS}[b x]^2 \, dx$$

$$\int x^3 \operatorname{FresnelS}[b x]^2 \, dx$$

$$\int \frac{\operatorname{FresnelS}[b x]^2}{x} \, dx$$

$$\int \frac{\operatorname{FresnelS}[b x]^2}{x} \, dx$$

$$\int \frac{\operatorname{FresnelS}[b x]^2}{x^5} \, dx$$

$$\int \frac{\operatorname{FresnelS}[b x]^2}{x^5} \, dx$$

$$\int \frac{\operatorname{FresnelS}[b x]^2}{x^9} \, dx$$

$$\int \frac{\operatorname{FresnelS}[b x]^2}{x^9} \, dx$$

- *Maple* is not able to integrate $x^m \operatorname{FresnelS}[b x]^2$ if m mod 4 equals 3:

```
int(x^7*FresnelS(b*x)^2, x);
```

$$\int x^7 \operatorname{FresnelS}(b x)^2 \, dx$$

```
int(x^3*FresnelS(b*x)^2, x);
```

$$\int x^3 \operatorname{FresnelS}(b x)^2 \, dx$$

```
int(FresnelS(b*x)^2/x, x);
```

$$\int \frac{\text{FresnelS}(b\,x)^2}{x}\,dx$$

```
int(FresnelS(b*x)^2/x^5, x);
```

$$\int \frac{\text{FresnelS}(b\,x)^2}{x^5}\,dx$$

```
int(FresnelS(b*x)^2/x^9, x);
```

$$\int \frac{\text{FresnelS}(b\,x)^2}{x^9}\,dx$$

Note that these systems give similar results to the above for the Fresnel cosine function.