Hyperbolic Function Integration Problem 1

$$\int x^m \operatorname{Tanh}[a+b\,x]\,dx$$

- *Rubi* returns m+2 term sums for positive integer m:

```
Int[x Tanh[a + b x], x]
```

$$-\frac{x^2}{2}+\frac{x\,\operatorname{Log}\left[1+e^{2a+2bx}\right]}{b}+\frac{\operatorname{PolyLog}\left[2,-e^{2a+2bx}\right]}{2\,b^2}$$

```
Int[x^2 Tanh[a + b x], x]
```

$$-\frac{x^3}{3}+\frac{x^2\,\operatorname{Log}\left[1+e^{2a+2bx}\right]}{b}+\frac{x\,\operatorname{PolyLog}\left[2,-e^{2a+2bx}\right]}{b^2}-\frac{\operatorname{PolyLog}\left[3,-e^{2a+2bx}\right]}{2\,b^3}$$

```
Int[x^3 Tanh[a + b x], x]
```

$$-\frac{x^4}{4}+\frac{x^3\,\operatorname{Log}\left[1+e^{2a+2bx}\right]}{b}+\frac{3\,x^2\,\operatorname{PolyLog}\left[2,-e^{2a+2bx}\right]}{2\,b^2}-\frac{3\,x\,\operatorname{PolyLog}\left[3,-e^{2a+2bx}\right]}{2\,b^3}+\frac{3\,\operatorname{PolyLog}\left[4,-e^{2a+2bx}\right]}{4\,b^4}$$

- *Mathematica* returns a 10 term sum involving the imaginary unit when m is 1:

$$\int x \operatorname{Tanh}[a+b\,x]\,dx$$

$$\frac{i\,\pi\,x}{2\,b}+\frac{x\,\operatorname{ArcTanh}[\operatorname{Coth}[a]]}{b}-\frac{i\,\pi\,\operatorname{Log}\left[1+e^{2bx}\right]}{2\,b^2}+\frac{x\,\operatorname{Log}\left[1-e^{-2\,(b\,x+\operatorname{ArcTanh}[\operatorname{Coth}[a]])}\right]}{b}+\frac{\operatorname{ArcTanh}[\operatorname{Coth}[a]]\,\operatorname{Log}\left[1-e^{-2\,(b\,x+\operatorname{ArcTanh}[\operatorname{Coth}[a]])}\right]}{b^2}+\frac{i\,\pi\,\operatorname{Log}[\operatorname{Cosh}[b\,x]]}{2\,b^2}-\frac{\operatorname{ArcTanh}[\operatorname{Coth}[a]]\,\operatorname{Log}[i\,\operatorname{Sinh}[b\,x+\operatorname{ArcTanh}[\operatorname{Coth}[a]]]]}{b^2}-\frac{\operatorname{PolyLog}\left[2,e^{-2\,(b\,x+\operatorname{ArcTanh}[\operatorname{Coth}[a]])}\right]}{2\,b^2}+\frac{1}{2}\,x^2\,\operatorname{Tanh}[a]-\frac{1}{2}\,e^{-\operatorname{ArcTanh}[\operatorname{Coth}[a]]}\,x^2\,\sqrt{-\operatorname{Csch}[a]^2}\,\operatorname{Tanh}[a]$$

$$\int x^2 \operatorname{Tanh}[a+b\,x]\,dx$$

$$-\frac{x^3}{3}+\frac{x^2\,\operatorname{Log}\left[1+e^{2\,(a+b\,x)}\right]}{b}+\frac{x\,\operatorname{PolyLog}\left[2,-e^{2\,(a+b\,x)}\right]}{b^2}-\frac{\operatorname{PolyLog}\left[3,-e^{2\,(a+b\,x)}\right]}{2\,b^3}$$

$$\int x^3 \operatorname{Tanh}[a+b\,x]\,dx$$

$$-\frac{x^4}{4}+\frac{x^3\,\operatorname{Log}\left[1+e^{2\,(a+b\,x)}\right]}{b}+\frac{3\,x^2\,\operatorname{PolyLog}\left[2,-e^{2\,(a+b\,x)}\right]}{2\,b^2}-\frac{3\,x\,\operatorname{PolyLog}\left[3,-e^{2\,(a+b\,x)}\right]}{2\,b^3}+\frac{3\,\operatorname{PolyLog}\left[4,-e^{2\,(a+b\,x)}\right]}{4\,b^4}$$

- *Maple* returns m+5 term sums, 3 of which are superfluous since their derivative is zero:

```
int(x*tanh(a+b*x), x);
```

$$-\frac{a^2}{b^2}-\frac{2\,a\,x}{b}-\frac{x^2}{2}+\frac{2\,a\,\mathrm{Log}\left[e^{a+b\,x}\right]}{b^2}+\frac{x\,\mathrm{Log}\left[1+e^{2\,a+2\,b\,x}\right]}{b}+\frac{\mathrm{PolyLog}\left[2,-e^{2\,a+2\,b\,x}\right]}{2\,b^2}$$

```
int (x^2 * tanh (a + b * x), x);
```

$$\frac{4\,a^3}{3\,b^3}+\frac{2\,a^2\,x}{b^2}-\frac{x^3}{3}-\frac{2\,a^2\,\mathrm{Log}\left[e^{a+b\,x}\right]}{b^3}+\frac{x^2\,\mathrm{Log}\left[1+e^{2\,a+2\,b\,x}\right]}{b}+\frac{x\,\mathrm{PolyLog}\left[2,-e^{2\,a+2\,b\,x}\right]}{b^2}-\frac{\mathrm{PolyLog}\left[3,-e^{2\,a+2\,b\,x}\right]}{2\,b^3}$$

```
int (x^3 * tanh (a + b * x), x);
```

$$\frac{3\,a^4}{2\,b^4}-\frac{2\,a^3\,x}{b^3}-\frac{x^4}{4}+\frac{2\,a^3\,\mathrm{Log}\left[e^{a+b\,x}\right]}{b^4}+\frac{x^3\,\mathrm{Log}\left[1+e^{2\,a+2\,b\,x}\right]}{b}+$$
$$\frac{3\,x^2\,\mathrm{PolyLog}\left[2,-e^{2\,a+2\,b\,x}\right]}{2\,b^2}-\frac{3\,x\,\mathrm{PolyLog}\left[3,-e^{2\,a+2\,b\,x}\right]}{2\,b^3}+\frac{3\,\mathrm{PolyLog}\left[4,-e^{2\,a+2\,b\,x}\right]}{4\,b^4}$$

Note that these systems give similar results to the above for the hyperbolic cotangent function.

Hyperbolic Function Integration Problem 2

$$\int \frac{x^m}{a + b\,\mathrm{Sinh}[x]}\, dx$$

- *Rubi* returns 2m+2 term sums for positive integer m:

$$\mathrm{Int}\left[\frac{x}{a + b\,\mathrm{Sinh}[x]}, x\right]$$

$$\frac{x\,\mathrm{Log}\left[1+\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} - \frac{x\,\mathrm{Log}\left[1+\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} + \frac{\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} - \frac{\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}$$

$$\mathrm{Int}\left[\frac{x^2}{a + b\,\mathrm{Sinh}[x]}, x\right]$$

$$\frac{x^2\,\mathrm{Log}\left[1+\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} - \frac{x^2\,\mathrm{Log}\left[1+\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} + \frac{2\,x\,\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} - \frac{2\,x\,\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} - \frac{2\,\mathrm{PolyLog}\left[3,-\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} + \frac{2\,\mathrm{PolyLog}\left[3,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}$$

$$\mathrm{Int}\left[\frac{x^3}{a + b\,\mathrm{Sinh}[x]}, x\right]$$

$$\frac{x^3\,\mathrm{Log}\left[1+\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} - \frac{x^3\,\mathrm{Log}\left[1+\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} + \frac{3\,x^2\,\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} - \frac{3\,x^2\,\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} - \frac{6\,x\,\mathrm{PolyLog}\left[3,-\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} + \frac{6\,x\,\mathrm{PolyLog}\left[3,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} + \frac{6\,\mathrm{PolyLog}\left[4,-\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}} - \frac{6\,\mathrm{PolyLog}\left[4,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}$$

- *Mathematica* returns a huge result involving the imaginary unit when m is 1:

$$\int \frac{x}{a + b\,\mathrm{Sinh}[x]}\, dx$$

$$-\frac{i \pi \operatorname{ArcTanh}\left[\frac{-b+a \operatorname{Tanh}\left[\frac{x}{2}\right]}{\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{1}{\sqrt{-a^2-b^2}}\left(2 \operatorname{ArcCos}\left[-\frac{i a}{b}\right] \operatorname{ArcTanh}\left[\frac{(a+i b) \operatorname{Cot}\left[\frac{1}{4}(\pi+2 i x)\right]}{\sqrt{-a^2-b^2}}\right]+\right.$$

$$(\pi-2 i x) \operatorname{ArcTanh}\left[\frac{(a-i b) \operatorname{Tan}\left[\frac{1}{4}(\pi+2 i x)\right]}{\sqrt{-a^2-b^2}}\right]-\left(\operatorname{ArcCos}\left[-\frac{i a}{b}\right]+2 i \operatorname{ArcTanh}\left[\frac{(a+i b) \operatorname{Cot}\left[\frac{1}{4}(\pi+2 i x)\right]}{\sqrt{-a^2-b^2}}\right]\right)$$

$$\operatorname{Log}\left[\frac{(i a+b)\left(a+i\left(b+\sqrt{-a^2-b^2}\right)\right)\left(-i+\operatorname{Cot}\left[\frac{1}{4}(\pi+2 i x)\right]\right)}{b\left(i a+b+i \sqrt{-a^2-b^2} \operatorname{Cot}\left[\frac{1}{4}(\pi+2 i x)\right]\right)}\right]-$$

$$\left(\operatorname{ArcCos}\left[-\frac{i a}{b}\right]-2 i \operatorname{ArcTanh}\left[\frac{(a+i b) \operatorname{Cot}\left[\frac{1}{4}(\pi+2 i x)\right]}{\sqrt{-a^2-b^2}}\right]\right)$$

$$\operatorname{Log}\left[\frac{(i a+b)\left(i a-b+\sqrt{-a^2-b^2}\right)\left(i+\operatorname{Cot}\left[\frac{1}{4}(\pi+2 i x)\right]\right)}{b\left(a-i b+\sqrt{-a^2-b^2} \operatorname{Cot}\left[\frac{1}{4}(\pi+2 i x)\right]\right)}\right]+$$

$$\left(\operatorname{ArcCos}\left[-\frac{i a}{b}\right]-2 i \operatorname{ArcTanh}\left[\frac{(a+i b) \operatorname{Cot}\left[\frac{1}{4}(\pi+2 i x)\right]}{\sqrt{-a^2-b^2}}\right]-2 i \operatorname{ArcTanh}\left[\frac{(a-i b) \operatorname{Tan}\left[\frac{1}{4}(\pi+2 i x)\right]}{\sqrt{-a^2-b^2}}\right]\right)$$

$$\operatorname{Log}\left[\frac{\left(\frac{1}{2}-\frac{i}{2}\right) \sqrt{-a^2-b^2}\, e^{-x/2}}{\sqrt{-i b} \sqrt{a+b \operatorname{Sinh}[x]}}\right]+$$

$$\left(\operatorname{ArcCos}\left[-\frac{i a}{b}\right]+2 i\left(\operatorname{ArcTanh}\left[\frac{(a+i b) \operatorname{Cot}\left[\frac{1}{4}(\pi+2 i x)\right]}{\sqrt{-a^2-b^2}}\right]+\operatorname{ArcTanh}\left[\frac{(a-i b) \operatorname{Tan}\left[\frac{1}{4}(\pi+2 i x)\right]}{\sqrt{-a^2-b^2}}\right]\right)\right)$$

$$\operatorname{Log}\left[\frac{\left(\frac{1}{2}+\frac{i}{2}\right) \sqrt{-a^2-b^2}\, e^{x/2}}{\sqrt{-i b} \sqrt{a+b \operatorname{Sinh}[x]}}\right]+i\left(\operatorname{PolyLog}\left[2, \frac{\left(i a+\sqrt{-a^2-b^2}\right)\left(i a+b-i \sqrt{-a^2-b^2} \operatorname{Cot}\left[\frac{1}{4}(\pi+2 i x)\right]\right)}{b\left(i a+b+i \sqrt{-a^2-b^2} \operatorname{Cot}\left[\frac{1}{4}(\pi+2 i x)\right]\right)}\right]-\right.$$

$$\left.\left.\operatorname{PolyLog}\left[2, \frac{\left(a+i \sqrt{-a^2-b^2}\right)\left(-a+i b+\sqrt{-a^2-b^2} \operatorname{Cot}\left[\frac{1}{4}(\pi+2 i x)\right]\right)}{b\left(i a+b+i \sqrt{-a^2-b^2} \operatorname{Cot}\left[\frac{1}{4}(\pi+2 i x)\right]\right)}\right]\right)\right)$$

$$\int \frac{x^2}{a+b \operatorname{Sinh}[x]} \, dx$$

$$\frac{x^2 \operatorname{Log}\left[1+\frac{b e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{x^2 \operatorname{Log}\left[1+\frac{b e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{2 x \operatorname{PolyLog}\left[2, \frac{b e^x}{-a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-$$

$$\frac{2 x \operatorname{PolyLog}\left[2,-\frac{b e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{2 \operatorname{PolyLog}\left[3, \frac{b e^x}{-a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{2 \operatorname{PolyLog}\left[3,-\frac{b e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}$$

$$\int \frac{x^3}{a+b \operatorname{Sinh}[x]} \, dx$$

$$\frac{x^3\,\mathrm{Log}\left[1+\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{x^3\,\mathrm{Log}\left[1+\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{3\,x^2\,\mathrm{PolyLog}\left[2,\frac{b\,e^x}{-a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{3\,x^2\,\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-$$

$$\frac{6\,x\,\mathrm{PolyLog}\left[3,\frac{b\,e^x}{-a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{6\,x\,\mathrm{PolyLog}\left[3,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{6\,\mathrm{PolyLog}\left[4,\frac{b\,e^x}{-a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{6\,\mathrm{PolyLog}\left[4,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}$$

- *Maple* is only able to integrate $\frac{x^m}{a+b\,\mathrm{Sinh}[x]}$ when m is 1:

```
int(x/(a+b*sinh(x)),x);
```

$$\frac{x\,\mathrm{Log}\left[1+\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{x\,\mathrm{Log}\left[1+\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}$$

```
int(x^2/(a+b*sinh(x)),x);
```

$$\int\frac{x^2}{a+b\,\sinh(x)}\,dx$$

```
int(x^3/(a+b*sinh(x)),x);
```

$$\int\frac{x^3}{a+b\,\sinh(x)}\,dx$$

Note that these systems give similar results to the above for the hyperbolic cosine function.

Hyperbolic Function Integration Problem 3

$$\int \mathbf{Sech}[a+b\,x]^4\,\mathbf{Tanh}[a+b\,x]^n\,dx$$

- *Rubi* maintains the symmetry between the trig and hyperbolic functions:

```
Int[Sec[a + b x]^4 Tan[a + b x]^n, x]
```

$$\frac{\mathrm{Tan}[a+b\,x]^{1+n}}{b\,(1+n)}+\frac{\mathrm{Tan}[a+b\,x]^{3+n}}{b\,(3+n)}$$

```
Int[Sech[a + b x]^4 Tanh[a + b x]^n, x]
```

$$\frac{\mathrm{Tanh}[a+b\,x]^{1+n}}{b\,(1+n)}-\frac{\mathrm{Tanh}[a+b\,x]^{3+n}}{b\,(3+n)}$$

- *Mathematica* is able to integrate the trig expression but not the corresponding hyperbolic one:

$$\int \mathrm{Sec}[a+b\,x]^4\,\mathrm{Tan}[a+b\,x]^n\,dx$$

$$\frac{(2+n+\mathrm{Cos}[2\,(a+b\,x)])\,\mathrm{Sec}[a+b\,x]^2\,\mathrm{Tan}[a+b\,x]^{1+n}}{b\,(1+n)\,(3+n)}$$

$$\int \mathrm{Sech}[a+b\,x]^4\,\mathrm{Tanh}[a+b\,x]^n\,dx$$

$$\int \mathrm{Sech}[a+b\,x]^4\,\mathrm{Tanh}[a+b\,x]^n\,dx$$

- *Maple* is unable to integrate the trig expression and returns a huge result for the hyperbolic one:

```
int(sec(a+b*x)^4*tan(a+b*x)^n, x);
```

$$\int \mathrm{Sec}[a+b\,x]^4\,\mathrm{Tan}[a+b\,x]^n\,dx$$

```
int(sech(a+b*x)^4*tanh(a+b*x)^n, x);
```

```
2 *
 (-3 * exp(2 * a + 2 * b * x) + exp(6 * a + 6 * b * x) + 3 * exp(4 * a + 4 * b * x) - 1 + 2 * exp(4 * a + 4 * b * x) * n - 2 * n *
        exp(2 * a + 2 * b * x)) / (1 + n) / b / (3 + n) / (exp(2 * a + 2 * b * x) + 1)^3 * exp
 (-1 / 2 * n * (-2 * ln(exp(a + b * x) - 1) - 2 * ln(1 + exp(a + b * x)) + 2 * ln(exp(2 * a + 2 * b * x) + 1) +
     I * Pi * csgn(I * (1 + exp(a + b * x)) / (exp(2 * a + 2 * b * x) + 1))^3 -
     I * Pi * csgn(I * (1 + exp(a + b * x)) / (exp(2 * a + 2 * b * x) + 1))^2 * csgn(I / (exp(2 * a + 2 * b * x) + 1)) -
     I * Pi * csgn(I * (1 + exp(a + b * x)) / (exp(2 * a + 2 * b * x) + 1))^2 * csgn(I * (1 + exp(a + b * x))) +
     I * Pi * csgn(I * (1 + exp(a + b * x)) / (exp(2 * a + 2 * b * x) + 1)) *
      csgn(I * (1 + exp(a + b * x))) * csgn(I / (exp(2 * a + 2 * b * x) + 1)) +
     I * Pi * csgn(I * (exp(a + b * x) - 1) / (exp(2 * a + 2 * b * x) + 1) * (1 + exp(a + b * x)))^3 -
     I * Pi * csgn(I * (exp(a + b * x) - 1) / (exp(2 * a + 2 * b * x) + 1) * (1 + exp(a + b * x)))^2 *
      csgn(I * (1 + exp(a + b * x)) / (exp(2 * a + 2 * b * x) + 1)) -
     I * Pi * csgn(I * (exp(a + b * x) - 1) / (exp(2 * a + 2 * b * x) + 1) * (1 + exp(a + b * x)))^2 *
      csgn(I * (exp(a + b * x) - 1)) +
     I * Pi * csgn(I * (exp(a + b * x) - 1) / (exp(2 * a + 2 * b * x) + 1) * (1 + exp(a + b * x))) * csgn
      (I * (exp(a + b * x) - 1)) * csgn(I * (1 + exp(a + b * x)) / (exp(2 * a + 2 * b * x) + 1))))
```

Hyperbolic Function Integration Problem 4

$$\int \text{Sinh}[x]\ \text{Sech}[n\ x]\, dx$$

- The *Rubi* results are simple, expressed in hyperbolic form and grow modestly with n:

```
Int[Sinh[x] Sech[x], x]
```

$$\text{Log}[\text{Cosh}[x]]$$

```
Int[Sinh[x] Sech[2 x], x]
```

$$-\frac{\text{ArcTanh}\left[\sqrt{2}\ \text{Cosh}[x]\right]}{\sqrt{2}}$$

```
Int[Sinh[x] Sech[3 x], x]
```

$$\frac{1}{3}\,\text{ArcTanh}\left[1-\frac{8\,\text{Cosh}[x]^2}{3}\right]$$

```
Int[Sinh[x] Sech[4 x], x]
```

$$\frac{1}{4}\sqrt{2+\sqrt{2}}\ \text{ArcTanh}\left[\frac{2\,\text{Cosh}[x]}{\sqrt{2-\sqrt{2}}}\right]-\frac{1}{4}\sqrt{2-\sqrt{2}}\ \text{ArcTanh}\left[\frac{2\,\text{Cosh}[x]}{\sqrt{2+\sqrt{2}}}\right]$$

- The *Mathematica* results grow unpredictably and is not in closed-form when n is 4:

$$\int \text{Sinh}[x]\ \text{Sech}[x]\, dx$$

$$\text{Log}[\text{Cosh}[x]]$$

$$\int \text{Sinh}[x]\ \text{Sech}[2\ x]\, dx$$

$$\frac{1}{4\sqrt{2}}\left(-2\,i\,\text{ArcTan}\left[\frac{\text{Cosh}\left[\frac{x}{2}\right]+\text{Sinh}\left[\frac{x}{2}\right]}{\left(1+\sqrt{2}\right)\text{Cosh}\left[\frac{x}{2}\right]-\left(-1+\sqrt{2}\right)\text{Sinh}\left[\frac{x}{2}\right]}\right]+2\,i\,\text{ArcTan}\left[\frac{\text{Cosh}\left[\frac{x}{2}\right]+\text{Sinh}\left[\frac{x}{2}\right]}{\left(-1+\sqrt{2}\right)\text{Cosh}\left[\frac{x}{2}\right]-\left(1+\sqrt{2}\right)\text{Sinh}\left[\frac{x}{2}\right]}\right]-\right.$$
$$\left.4\,\text{ArcTanh}\left[\sqrt{2}-i\,\text{Tanh}\left[\frac{x}{2}\right]\right]-\text{Log}\left[2\left(\sqrt{2}+2\,\text{Cosh}[x]\right)\right]+\text{Log}\left[-2\sqrt{2}+4\,\text{Cosh}[x]\right]\right)$$

$$\int \text{Sinh}[x]\ \text{Sech}[3\ x]\, dx$$

$$-\frac{1}{3}\,\text{Log}[\text{Cosh}[x]]+\frac{1}{6}\,\text{Log}[-1+2\,\text{Cosh}[2\ x]]$$

```
∫Sinh[x] Sech[4 x]ⅆx
```

$$\frac{1}{16}\,\text{RootSum}\Big[1+\#1^8\,\&,\ \frac{1}{\#1^5}\Big(-x-2\,\text{Log}\Big[-\text{Cosh}\Big[\frac{x}{2}\Big]-\text{Sinh}\Big[\frac{x}{2}\Big]+\text{Cosh}\Big[\frac{x}{2}\Big]\#1-\text{Sinh}\Big[\frac{x}{2}\Big]\#1\Big]+x\,\#1^2+2\,\text{Log}\Big[-\text{Cosh}\Big[\frac{x}{2}\Big]-\text{Sinh}\Big[\frac{x}{2}\Big]+\text{Cosh}\Big[\frac{x}{2}\Big]\#1-\text{Sinh}\Big[\frac{x}{2}\Big]\#1\Big]\#1^2\Big)\,\&\Big]$$

- The *Maple* results are simple, but expressed in exponential form and not in closed-form when n is 4:

```
int(sinh(x)*sech(x), x);
```

Log[Cosh[x]]

```
int(sinh(x)*sech(2*x), x);
```

$$\frac{\text{Log}\left[1-\sqrt{2}\,e^{x}+e^{2x}\right]}{2\sqrt{2}}-\frac{\text{Log}\left[1+\sqrt{2}\,e^{x}+e^{2x}\right]}{2\sqrt{2}}$$

```
int(sinh(x)*sech(3*x), x);
```

$$-\frac{1}{3}\,\text{Log}\left[1+e^{2x}\right]+\frac{1}{6}\,\text{Log}\left[1-e^{2x}+e^{4x}\right]$$

```
int(sinh(x)*sech(4*x), x);
```

2 * sum (_R * ln (exp (2 * x) + (4096 * _R^3 - 48 * _R) * exp (x) + 1), _R = RootOf (32 768 * _Z^4 - 512 * _Z^2 + 1))

Note that these systems give similar results to the above for the hyperbolic cosine function.

Hyperbolic Function Integration Problem 5

$$\int\sqrt{a+b\,\mathrm{Tanh}[x]}\,dx \quad \& \quad \int\sqrt{a+b\,\mathrm{Coth}[x]}\,dx$$

- The *Rubi* results are simple and symmetric:

```
{Int[Sqrt[1 + Tanh[x]], x], Int[Sqrt[1 + Coth[x]], x]}
```

$$\left\{\sqrt{2}\,\mathrm{ArcTanh}\left[\frac{\sqrt{1+\mathrm{Tanh}[x]}}{\sqrt{2}}\right],\ \sqrt{2}\,\mathrm{ArcCoth}\left[\frac{\sqrt{1+\mathrm{Coth}[x]}}{\sqrt{2}}\right]\right\}$$

```
{Int[Sqrt[a + b Tanh[x]], x], Int[Sqrt[a + b Coth[x]], x]}
```

$$\left\{-\sqrt{a-b}\,\mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Tanh}[x]}}{\sqrt{a-b}}\right]+\sqrt{a+b}\,\mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Tanh}[x]}}{\sqrt{a+b}}\right],\right.$$
$$\left.-\sqrt{a-b}\,\mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Coth}[x]}}{\sqrt{a-b}}\right]+\sqrt{a+b}\,\mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Coth}[x]}}{\sqrt{a+b}}\right]\right\}$$

- The *Mathematica* results are more complicated involving the imaginary unit and not symmetric:

$$\left\{\int\sqrt{1+\mathrm{Tanh}[x]}\,dx,\ \int\sqrt{1+\mathrm{Coth}[x]}\,dx\right\}$$

$$\left\{\sqrt{2}\,\mathrm{ArcTanh}\left[\frac{\sqrt{1+\mathrm{Tanh}[x]}}{\sqrt{2}}\right],\ \frac{(1+i)\,\mathrm{ArcTan}\left[\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{i\,(1+\mathrm{Coth}[x])}\right](1+\mathrm{Coth}[x])^{3/2}}{(i\,(1+\mathrm{Coth}[x]))^{3/2}}\right\}$$

$$\left\{\int\sqrt{a+b\,\mathrm{Tanh}[x]}\,dx,\ \int\sqrt{a+b\,\mathrm{Coth}[x]}\,dx\right\}$$

$$\left\{-\sqrt{a-b}\,\mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Tanh}[x]}}{\sqrt{a-b}}\right]+\sqrt{a+b}\,\mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Tanh}[x]}}{\sqrt{a+b}}\right],\right.$$
$$\left.\frac{\left(-\sqrt{i\,(a-b)}\,\mathrm{ArcTanh}\left[\frac{\sqrt{i\,(a+b\,\mathrm{Coth}[x])}}{\sqrt{i\,(a-b)}}\right]+\sqrt{i\,(a+b)}\,\mathrm{ArcTanh}\left[\frac{\sqrt{i\,(a+b\,\mathrm{Coth}[x])}}{\sqrt{i\,(a+b)}}\right]\right)\sqrt{a+b\,\mathrm{Coth}[x]}}{\sqrt{i\,(a+b\,\mathrm{Coth}[x])}}\right\}$$

- The *Maple* results are simple and symmetric:

```
[int(sqrt(1+tanh(x)),x), int(sqrt(1+coth(x)),x)];
```

$$\left\{\sqrt{2}\,\mathrm{ArcTanh}\left[\frac{\sqrt{1+\mathrm{Tanh}[x]}}{\sqrt{2}}\right],\ \sqrt{2}\,\mathrm{ArcCoth}\left[\frac{\sqrt{1+\mathrm{Coth}[x]}}{\sqrt{2}}\right]\right\}$$

```
[int(sqrt(a+b*tanh(x)),x), int(sqrt(a+b*coth(x)),x)];
```

$$\left\{-\sqrt{a-b}\ \text{ArcTanh}\left[\frac{\sqrt{a+b\ \text{Tanh}[x]}}{\sqrt{a-b}}\right]+\sqrt{a+b}\ \text{ArcTanh}\left[\frac{\sqrt{a+b\ \text{Tanh}[x]}}{\sqrt{a+b}}\right],\right.$$

$$\left.-\sqrt{a-b}\ \text{ArcTanh}\left[\frac{\sqrt{a+b\ \text{Coth}[x]}}{\sqrt{a-b}}\right]+\sqrt{a+b}\ \text{ArcTanh}\left[\frac{\sqrt{a+b\ \text{Coth}[x]}}{\sqrt{a+b}}\right]\right\}$$

Hyperbolic Function Integration Problem 6

$$\int \frac{\text{Sech}[x]^2}{\sqrt{a - b\,\text{Tanh}[x]^2}}\, dx$$

- The *Rubi* results are simple for both symbolic and numeric a and b:

$$\text{Int}\left[\frac{\text{Sech}[x]^2}{\sqrt{a - b\,\text{Tanh}[x]^2}}, x\right]$$

$$-\frac{\text{ArcTan}\left[\frac{\text{Coth}[x]\sqrt{a-b\,\text{Tanh}[x]^2}}{\sqrt{b}}\right]}{\sqrt{b}}$$

$$\text{Int}\left[\frac{\text{Sech}[x]^2}{\sqrt{1 - b\,\text{Tanh}[x]^2}}, x\right]$$

$$\frac{\text{ArcSin}\left[\sqrt{b}\,\text{Tanh}[x]\right]}{\sqrt{b}}$$

$$\text{Int}\left[\frac{\text{Sech}[x]^2}{\sqrt{1 - 4\,\text{Tanh}[x]^2}}, x\right]$$

$$\frac{1}{2}\,\text{ArcSin}[2\,\text{Tanh}[x]]$$

- *Mathematica* apparently uses the hyperbolic tangent theta/2 substitution for the numeric case:

$$\int \frac{\text{Sech}[x]^2}{\sqrt{a - b\,\text{Tanh}[x]^2}}\, dx$$

$$\frac{\text{ArcTan}\left[\frac{\sqrt{2}\sqrt{b}\,\text{Sinh}[x]}{\sqrt{a+b+(a-b)\,\text{Cosh}[2x]}}\right]\sqrt{a+b+(a-b)\,\text{Cosh}[2x]}\;\text{Sech}[x]}{\sqrt{2}\sqrt{b}\sqrt{a-b\,\text{Tanh}[x]^2}}$$

$$\int \frac{\text{Sech}[x]^2}{\sqrt{1 - b\,\text{Tanh}[x]^2}}\, dx$$

$$\frac{\text{ArcSin}\left[\sqrt{b}\,\text{Tanh}[x]\right]}{\sqrt{b}}$$

$$\int \frac{\text{Sech}[x]^2}{\sqrt{1 - 4\,\text{Tanh}[x]^2}}\, dx$$

$$-\frac{1}{\sqrt{1-4\,\mathrm{Tanh}[x]^2}}\,2\,\mathrm{Cosh}\left[\frac{x}{2}\right]^2\left(\mathrm{EllipticF}\left[\mathrm{ArcSin}\left[\frac{\mathrm{Tanh}\left[\frac{x}{2}\right]}{\sqrt{7-4\sqrt{3}}}\right],\,97-56\sqrt{3}\right]+\right.$$

$$\left.2\,\mathrm{EllipticPi}\left[-7+4\sqrt{3},\,-\mathrm{ArcSin}\left[\frac{\mathrm{Tanh}\left[\frac{x}{2}\right]}{\sqrt{7-4\sqrt{3}}}\right],\,97-56\sqrt{3}\right]\right)$$

$$\mathrm{Sech}[x]\,\sqrt{7-4\sqrt{3}-\mathrm{Tanh}\left[\frac{x}{2}\right]^2}\,\sqrt{1+\left(-7+4\sqrt{3}\right)\mathrm{Tanh}\left[\frac{x}{2}\right]^2}$$

- *Maple* is unable to integrate $\frac{\mathbf{Sech[x]^2}}{\sqrt{\mathbf{a-b\,Tanh[x]^2}}}$ for symbolic and numeric variables a and b:

```
int(sech(x)^2/sqrt(a-b*tanh(x)^2), x);
```

$$\int \frac{\mathrm{Sech}[x]^2}{\sqrt{a-b\,\mathrm{Tanh}[x]^2}}\,dx$$

```
int(sech(x)^2/sqrt(1-b*tanh(x)^2), x);
```

$$\int \frac{\mathrm{Sech}[x]^2}{\sqrt{1-b\,\mathrm{Tanh}[x]^2}}\,dx$$

```
int(sech(x)^2/sqrt(1-4*tanh(x)^2), x);
```

$$\int \frac{\mathrm{Sech}[x]^2}{\sqrt{1-4\,\mathrm{Tanh}[x]^2}}\,dx$$

Hyperbolic Function Integration Problem 7

$$\int \frac{\text{Tanh}[x]}{\sqrt{a + b\,\text{Tanh}[x]^4}}\,dx$$

- *Rubi* is able to integrate the expression:

$$\text{Int}\left[\frac{\text{Tanh}[x]}{\sqrt{a + b\,\text{Tanh}[x]^4}}, x\right]$$

$$\frac{\text{ArcTanh}\left[\frac{\sqrt{a+b}\,\sqrt{a+b\,\text{Tanh}[x]^4}}{a+b\,\text{Tanh}[x]^2}\right]}{2\sqrt{a+b}}$$

- *Mathematica* is unable to integrate the expression:

$$\int \frac{\text{Tanh}[x]}{\sqrt{a + b\,\text{Tanh}[x]^4}}\,dx$$

$$\int \frac{\text{Tanh}[x]}{\sqrt{a + b\,\text{Tanh}[x]^4}}\,dx$$

- *Maple* is able to integrate the expression:

```
int(tanh(x)/sqrt(a+b*tanh(x)^4),x);
```

$$\frac{\text{ArcTanh}\left[\frac{a+b\,\text{Tanh}[x]^2}{\sqrt{a+b}\,\sqrt{a+b\,\text{Tanh}[x]^4}}\right]}{2\sqrt{a+b}}$$

Hyperbolic Function Integration Problem 8

$$\int \frac{\sqrt{\text{Sinh}[a+b\,x]}}{\sqrt{\text{Cosh}[a+b\,x]}}\,dx$$

- The *Rubi* results are symmetric and involve only elementary functions:

```
Int[Sqrt[Sinh[a + b x]/Cosh[a + b x]], x]
```

$$-\frac{\text{ArcTan}\left[\sqrt{\text{Tanh}[a+b\,x]}\right]}{b}+\frac{\text{ArcTanh}\left[\sqrt{\text{Tanh}[a+b\,x]}\right]}{b}$$

```
Int[Sqrt[Sinh[a + b x]]/Sqrt[Cosh[a + b x]], x]
```

$$-\frac{\text{ArcTan}\left[\frac{\sqrt{\text{Sinh}[a+b\,x]}}{\sqrt{\text{Cosh}[a+b\,x]}}\right]}{b}+\frac{\text{ArcTanh}\left[\frac{\sqrt{\text{Sinh}[a+b\,x]}}{\sqrt{\text{Cosh}[a+b\,x]}}\right]}{b}$$

- The *Mathematica* results are not symmetric and involve a hypergeometric function:

$$\int \sqrt{\frac{\text{Sinh}[a+b\,x]}{\text{Cosh}[a+b\,x]}}\,dx$$

$$-\frac{\text{ArcTan}\left[\sqrt{\text{Tanh}[a+b\,x]}\right]}{b}-\frac{\text{Log}\left[-1+\sqrt{\text{Tanh}[a+b\,x]}\right]}{2\,b}+\frac{\text{Log}\left[1+\sqrt{\text{Tanh}[a+b\,x]}\right]}{2\,b}$$

$$\int \frac{\sqrt{\text{Sinh}[a+b\,x]}}{\sqrt{\text{Cosh}[a+b\,x]}}\,dx$$

$$-\frac{2\sqrt{\text{Cosh}[a+b\,x]}\ \text{Hypergeometric2F1}\left[\frac{1}{4},\frac{1}{4},\frac{5}{4},\text{Cosh}[a+b\,x]^2\right]\text{Sinh}[a+b\,x]^{3/2}}{b\left(-\text{Sinh}[a+b\,x]^2\right)^{3/4}}$$

- *Maple* is able to integrate $\sqrt{\text{Tanh}[a+b\,x]}$ but not $\sqrt{\text{Sinh}[a+b\,x]\,/\,\text{Cosh}[a+b\,x]}$:

```
int(sqrt(tanh(a+b*x)),x);
```

$$-\frac{\text{ArcTan}\left[\sqrt{\text{Tanh}[a+b\,x]}\right]}{b}-\frac{\text{Log}\left[-1+\sqrt{\text{Tanh}[a+b\,x]}\right]}{2\,b}+\frac{\text{Log}\left[1+\sqrt{\text{Tanh}[a+b\,x]}\right]}{2\,b}$$

```
int(sqrt(sinh(a+b*x)/cosh(a+b*x)),x);
```

$$\int \sqrt{\frac{\text{Sinh}[a+b\,x]}{\text{Cosh}[a+b\,x]}}\,dx$$

The *Maple* result involves complex exponentials and an elliptic integral function:

```
int (sqrt (sinh (a + b * x)) / sqrt (cosh (a + b * x)), x);
```

```
2 / 3 / b * exp (a + b * x) * (exp (a + b * x) ^2 + 1) / ((exp (a + b * x) ^2 + 1) * exp (a + b * x)) ^ (1 / 2) -
 4 / 3 * I / b * (-I * (I + exp (a + b * x))) ^ (1 / 2) * 2^ (1 / 2) * (I * (-I + exp (a + b * x))) ^ (1 / 2) *
  (I * exp (a + b * x)) ^ (1 / 2) / (exp (a + b * x) + exp (a + b * x) ^3) ^ (1 / 2) *
  EllipticF ((-I * (I + exp (a + b * x))) ^ (1 / 2), 1 / 2 * 2^ (1 / 2))
```